Presence of minimal components in a Morse form foliation

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Abstract

Conditions and a criterion for the presence of minimal components in the foliation of a Morse form ω on a smooth closed oriented manifold M are given in terms of (1) the maximum rank of a subgroup in $H^1(M,\mathbb{Z})$ with trivial cup-product, (2) $\ker[\omega]$, and (3) $\operatorname{rk}\omega \overset{\text{def}}{=} \operatorname{rk}\operatorname{im}[\omega]$, where $[\omega]$ is the integration map.

Key words: Morse form foliation, minimal components, form rank, cup-product
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1 Introduction

Let M be a connected smooth closed oriented n-dimensional manifold and ω a Morse form on M, i.e. a closed 1-form with Morse singularities (locally the differential of a Morse function). This form defines a foliation $\mathcal{F}_\omega$ on $M \setminus \operatorname{Sing}\omega$, where $\operatorname{Sing}\omega$ are the form's singularities.

The problem of studying the topology of such foliations was set up by S. Novikov [9] as far back as in early 80s in connection with their numerous applications in physics [10,11], which have been recently impulsed by the new advances in the mathematical theory [2,3].

The topology of a Morse form foliation can be described as follows. Its leaves are either compact, non-compact compactifiable, or non-compactifiable. A leaf γ is called *compactifiable* if $\gamma \cup \operatorname{Sing}\omega$ is compact. There is a finite number of non-compact compactifiable leaves; thus their union together with $\operatorname{Sing}\omega$ has zero measure. The rest of M consists of a finite number of open areas covered by compact leaves (called *maximal components*) or non-compactifiable leaves (called *minimal components*).

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Compact leaves have neat properties [8]. All leaves in a maximal component are diffeomorphic. A maximal component is an open cylinder over any its leaf. The form's integral by any cycle lying in a maximal component is zero.

Non-compactifiable leaves, on the contrary, have very complex behaviour [1]. Each such leaf is dense in its minimal component. A minimal component can cover a rather complex set in M; for any M with Betti number $\beta_1(M) \geq 2$ there exists a foliation whose only minimal component covers the whole $M \setminus \operatorname{Sing}\omega$. A minimal component contains at least two homologically independent cycles with non-commensurable integrals [8].

In this paper we consider conditions for a foliation to have minimal components.

The form's singularities give little information on the foliation topology. $\mathcal{F}_\omega$ is compact (i.e., all its leaves are compact) if and only if all singularities of ω are spherical. Otherwise there always exists a form with the same singularities of the same indices but with the foliation without minimal components [12].

A more useful characteristic of the form is its *rank* $\operatorname{rk}\omega \stackrel{\text{def}}{=} \operatorname{rk}\operatorname{im}[\omega]$, where $[\omega](z) = \int_z \omega \in \mathbb{R}$, i.e. the rank of its group of periods; it is a cohomologous invariant. If $\operatorname{rk}\omega \leq 1$, the foliation has no minimal components [9]. For $\operatorname{rk}\omega \geq 2$, the foliation of a non-singular form is minimal and uniquely ergodic; however, for forms with singularities the situation is much more complicated.

In any cohomology class with $\operatorname{rk}\omega \geq 2$ there is a form with a minimal foliation [1]. If the cohomology class of ω, $\operatorname{rk}\omega \geq 2$, contains a non-singular form, then $\mathcal{F}_\omega$ has a minimal component, though—unlike non-singular case—it is not necessarily minimal [4]. Existence of non-singular form in a given cohomology class was studied in [5]; however, the only manifolds allowing non-singular closed forms are bundles over S^1 [13].

We show that for large enough $\operatorname{rk}\omega$ any foliation has a minimal component—namely, for $\operatorname{rk}\omega > h(M)$, where $h(M)$ is the maximum rank of an *isotropic* (i.e., with trivial cup-product) subgroup in $H^1(M,\mathbb{Z})$ (Theorem 13). In particular, the foliation of a Morse form in general position on a manifold with non-trivial cup-product has a minimal component (Theorem 18).

The mentioned Theorem 13 gives a simple yet powerful practical sufficient condition for the presence of minimal components. Methods of calculating $h(M)$ for many important manifolds can be found in [7]; the most useful of them are listed in Remark 14. For example, $\mathcal{F}_\omega$ on M_g^2 with $\operatorname{rk}\omega > g = h(M_g^2)$ has a minimal component (Example 16), so does $\mathcal{F}_\omega$ on T^n (torus) with $\operatorname{rk}\omega > 1 = h(T^n)$ (Example 15).

Yet the group $\ker[\omega]$ gives more fine-grained information on the foliation structure than the mere $\operatorname{rk}\omega = \operatorname{rk}\operatorname{im}[\omega]$. We call a subgroup $G \subseteq H_1(M)$ *parallel* if there exists an isotropic subgroup $H \subseteq H^1(M,\mathbb{Z})$ such that any homomorphism $\varphi : G \to \mathbb{Z}$ is realized by some element of H. If any of the following equivalent conditions holds then $\mathcal{F}_\omega$ has a minimal component (Theorem 11):

(i) For any parallel subgroup G it holds $\operatorname{rk} G - \operatorname{rk}(G \cap \ker[\omega]) < \operatorname{rk}\omega$ (note that non-strict inequality here holds for any group).
(ii) The same holds for any parallel subgroup G such that $G \cap \ker[\omega] = 0$.
(iii) The same holds for any maximal parallel subgroup G.

Finally, the foliation $\mathcal{F}_\omega$ has a minimal component if and only if there exists $z \in H_1(M) \setminus \ker[\omega]$ such that $z \circ [\gamma_i] = 0$ (intersection index) for all (compact) leaves $\gamma_1, \ldots, \gamma_{M(\omega)}$, one from each maximal component (Theorem 7).

Note that cohomologous invariants of ω alone do not give much information on the presence of minimal components, especially when it comes to necessary conditions (for any form with $\operatorname{rk}\omega \geq 2$ there is a cohomologous form with minimal foliation [1]). So we had to bring into consideration some characteristics of the manifold ($h(M)$, parallel subgroups) and the foliation (γ_i).

The paper is organized as follows. Section 2 introduces some definitions and facts connected with Morse form foliation. Auxiliary Section 3 is devoted to expressing $H_1(M)$ in terms of the foliation structure. In Section 4 we give a criterion (Theorem 7) and a necessary condition for a foliation to have a minimal component in terms of $\ker[\omega]$. Finally, in Section 5 we give sufficient conditions for a foliation to have a minimal component in terms of $\ker[\omega]$ (Theorem 11), $h(M)$ (Theorem 13), and cup-product (Theorem 18).

2 A Morse form foliation

In this section we introduce, for future reference, some useful notions and facts about Morse forms and their foliations.

Recall that M is a connected smooth closed oriented n-dimensional manifold; $n \geq 2$. A closed 1-form ω on M is called a *Morse form* if it is locally the differential of a Morse function. $\operatorname{Sing}\omega = \{p \in M \mid \omega(p) = 0\}$ denotes the set of its singularities; this set is finite since the singularities are isolated and M is compact. On $M \setminus \operatorname{Sing}\omega$ the form defines a foliation $\mathcal{F}_\omega$.

Definition 1 *A leaf $\gamma \in \mathcal{F}_\omega$ is called* compactifiable *if $\gamma \cup \operatorname{Sing}\omega$ is compact; otherwise it is called* non-compactifiable*.*

Note that a compact leaf is compactifiable. The number $K(\omega)$ of non-compact compactifiable leaves γ_i^0 is finite and can be estimated in terms of the number of singularities of ω [8].

Definition 2 *A connected component $\mathcal{C}$ of the union of compact leaves is called* maximal component *of the foliation.*

A maximal component is open; the number $M(\omega)$ of maximal components is finite and can be estimated in terms of homological characteristics of M and the number of singularities of ω [8].

Consider the following decomposition into mutually disjoint sets holds:

$$M = \Big(\bigcup_{i=1}^{M(\omega)} \mathcal{C}_i\Big) \cup \Delta, \tag{1}$$

where $\mathcal{C}_i$ are all maximal components and

$$\Delta = \Big(\bigcup_{i=1}^{m(\omega)} C_i^{min}\Big) \cup \Big(\bigcup_{i=1}^{K(\omega)} \gamma_i^0\Big) \cup \operatorname{Sing}\omega, \tag{2}$$

C_i^{min} being all minimal components of $\mathcal{F}_\omega$ and $m(\omega)$ being their number. The closed set Δ has a finite number of connected components Δ_j.

If $\operatorname{Sing}\omega = \emptyset$ then $\mathcal{F}_\omega$ is either minimal or compact. In the latter case it has exactly one maximal component $\mathcal{C} = M$, which is a bundle over S^1 with fiber $\gamma \in \mathcal{F}_\omega$ [13].

In the rest of this paper we suppose $\operatorname{Sing}\omega \neq \emptyset$. In this case each maximal component $\mathcal{C}_i$ is a cylinder over a compact leaf:

$$\mathcal{C}_i \cong \gamma_i \times (0,1), \tag{3}$$

where the diffeomorphism maps γ_i to leaves of $\mathcal{F}_\omega$; this map can be continuously extended to $\gamma_i \times [0,1]$ [8]. Since $\partial\mathcal{C}_i \subseteq \Delta$ consists of one or two connected components, each $\mathcal{C}_i$ adjoints one or two of Δ_j. Therefore the decomposition (1) allows representing M as the *foliation graph* Γ—a connected pseudograph (a graph admitting multiple loops and edges) with edges $\mathcal{C}_i$ and vertices Δ_j; an edge $\mathcal{C}_i$ is incident to a vertex Δ_j if $\partial\mathcal{C}_i \cap \Delta_j \neq \emptyset$; see Figure 1.

Definition 3 *The group H_ω generated by the homology classes of all compact leaves is called the homology group of the foliation.*

Since M is closed and oriented, the group $H_{n-1}(M)$ is finitely generated and free; therefore so is $H_\omega \subseteq H_{n-1}(M)$.

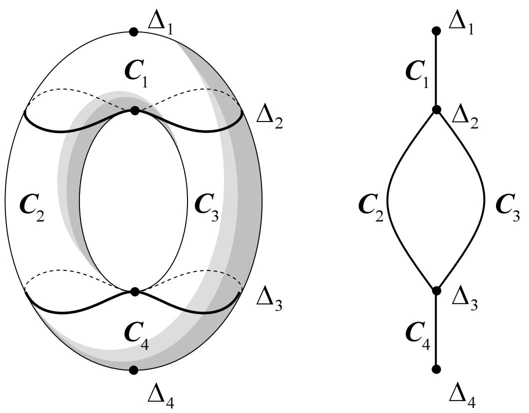

Fig. 1. Decomposition (1) and the corresponding foliation graph.

A set of elements generating a free group might not contain its basis, e.g., $\mathbb{Z} = \langle 2, 3 \rangle$. However:

Theorem 4 *In H_ω there exists a basis e consisting of homology classes of leaves: $e = \{[\gamma_1], \ldots, [\gamma_m]\}$, $\gamma_i \in \mathcal{F}_\omega$.*

PROOF. Consider a spanning tree T of Γ and the corresponding chords $h_1, \ldots, h_m$. We will show that $e = \{[\gamma_1], \ldots, [\gamma_m]\}$ is the desired basis, where γ_i is any leaf in the maximal component $h_i = \gamma_i \times (0, 1)$ (all leaves in a maximal component are homologous).

(i) The system e is independent. Indeed, let z be a cycle in the foliation graph Γ:

$$z = (p_1, x_1, \ldots, p_s, x_s, p_{s+1}), \quad p_{s+1} = p_1,$$

where $x_i \neq x_j$ are edges connecting vertices p_i, p_{i+1}. For z, a closed curve α in M can be (non-uniquely) constructed from the elements of the cylinders $x_i = \gamma_i \times (0, 1)$ connected by segments lying in $p_i = \Delta_i$; obviously $[\alpha] \circ [\gamma_i] = 1$.

For the chords $h_1, \ldots, h_m$ a system of cycles $z_1, \ldots, z_m$ in Γ can be constructed such that each h_i belongs to exactly one cycle z_i; denote $\alpha_1, \ldots, \alpha_m$ the corresponding closed curves in M. Then given $\sum_i n_i[\gamma_i] = 0$, for any j it holds $0 = [\alpha_j] \circ \sum_i n_i[\gamma_i] = n_j$.

(ii) $\langle e \rangle = H_\omega$. Indeed, consider a leaf γ such that its maximal component $x \notin \{h_i\}$. Then $x \in T$ is a bridge connecting two different (non-empty) connected components: $T - x = T' \cup T''$, i.e. $\Gamma - (x \cup \{h_i\}) = T' \cup T''$. The latter means that $\gamma \cup \{\gamma_i\}$ separate the two corresponding submanifolds in M, i.e. $[\gamma] + \sum_{i \in I} \pm[\gamma_i] = 0$. $\square$

In fact from the proof it follows that for every compact leaf γ, the coordinates of $[\gamma]$ in the basis e belong to $\{\pm 1, 0\}$.

3 The manifold's homologies and the foliation

Recall that $\mathcal{C}_k = \gamma_k \times (0,1)$, $k = 1, ..., M(\omega)$, are all maximal components and $\Delta = M \setminus (\bigcup_k \mathcal{C}_k)$. We will study the relationship between $H_1(M)$ and the decomposition (1).

Theorem 5 *Let $z \in H_1(M)$. If $z \circ [\gamma_k] = 0$ for all $k = 1, \dots, M(\omega)$ then $z \in i_* H_1(\Delta)$, where $i : \Delta \hookrightarrow M$.*

PROOF. Let $\varphi_k : \gamma_k \times I \to M$, $I = (-1,1)$ be the diffeomorphisms from (3), with $\gamma_k = \varphi_k(\gamma_k, 0) \subset M$.

Below we will show that z is realized by a closed curve that does not intersect with any γ_k. Given this, consider $M' = M \setminus (\bigcup_k \gamma_k)$; $z \in j_* H_1(M')$, $j : M' \hookrightarrow M$. By (1),

$$M' = \Delta \cup \left(\bigcup_k \varphi_k \left(\gamma_k \times (-1,0) \right) \cup \varphi_k \left(\gamma_k \times (0,1) \right) \right).$$

Thus Δ is the deformation retract of M', the corresponding homotopy on $M' \setminus \Delta$ being $r_s \left(\varphi_k(x \times t) \right) = \varphi_k \left(x \times (s + (1 \pm s)t) \right)$; recall that φ_k can be continuously extended to $\gamma_k \times [-1,1]$ with $\gamma_k \times \{\pm 1\} \subseteq \Delta$. This proves the theorem.

It remains to show that z can be realized by a curve that does not intersect with any γ_k. Denote $\gamma = \gamma_k$ and $\varphi = \varphi_k$. Let the orientation of γ be such that $\varphi(x,t)$ goes along its normal vector as t increases.

Consider a closed curve α realizing z, see Figure 2. Without loss of generality we can assume that α is transverse to $\gamma = \gamma_k$ and even that in a small enough neighborhood $U(\gamma)$ it goes along the element I of the cylinder $\operatorname{im} \varphi$.

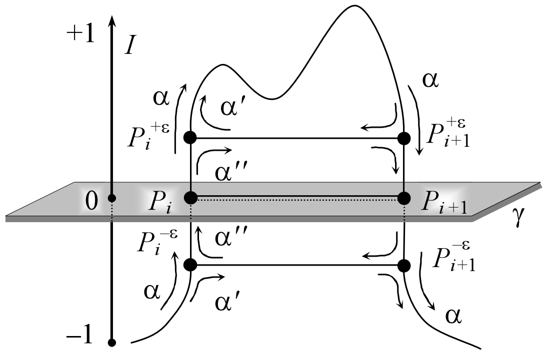

Fig. 2. Removing intersection points of α and γ.

Since $[\alpha] \circ [\gamma] = 0$, it holds $\alpha \cap \gamma = \cup_{i=1}^{2p} P_i$, where $\sum \operatorname{sgn} P_i = 0$. Suppose $p \neq 0$. Consider P_i, P_{i+1} such that $\operatorname{sgn} P_i \neq \operatorname{sgn} P_{i+1}$ and let $P_i^{-\varepsilon}, P_{i+1}^{-\varepsilon}$; $P_i^{+\varepsilon}, P_{i+1}^{+\varepsilon} \in$

$U(\gamma)\cap\alpha$, where $P_j^t = \varphi(P_j, t)$. Since γ is connected, there is a curve $P_iP_{i+1} \subset \gamma$. Obviously, $[\alpha] = [\alpha'] + [\alpha'']$, where

$$\alpha' = \Big(\alpha \setminus (P_i^{-\varepsilon}P_i^{+\varepsilon} \cup P_{i+1}^{+\varepsilon}P_{i+1}^{-\varepsilon})\Big) \cup P_i^{+\varepsilon}P_{i+1}^{+\varepsilon} \cup P_{i+1}^{-\varepsilon}P_i^{-\varepsilon}$$

and

$$\alpha'' = P_i^{-\varepsilon}P_i^{+\varepsilon}P_{i+1}^{+\varepsilon}P_{i+1}^{-\varepsilon};$$

here $P_i^{+\varepsilon}P_{i+1}^{+\varepsilon} = \varphi(P_iP_{i+1}, +\varepsilon)$ and $P_{i+1}^{-\varepsilon}P_i^{-\varepsilon} = -\varphi(P_iP_{i+1}, -\varepsilon)$. However, $[\alpha''] = 0$ since α'' is homotopy-equivalent to P_iP_{i+1}.

The new curve α' has $2p-2$ intersection points with $\gamma = \gamma_k$. Induction by p and then by k finishes the proof. $\square$

Theorem 6 *Let $e = \{[\gamma_1], \ldots, [\gamma_m]\}$, $\gamma_i \in \mathcal{F}_\omega$, be a basis of $H_\omega \subseteq H_{n-1}(M)$, $De = \{D[\gamma_1], \ldots, D[\gamma_m]\} \subset H_1(M)$ a system of dual cycles, i.e. $[\gamma_i] \circ D[\gamma_j] = \delta_{ij}$, and $DH_\omega = \langle De\rangle$. Then*

$$H_1(M) = \langle DH_\omega, i_*H_1(\Delta)\rangle.$$

Existence of e follows from Theorem 4.

PROOF. Let $z \in H_1(M)$ and $n_i = z \circ [\gamma_i]$. Consider the cycle $z' = z - \sum n_i D[\gamma_i]$. Then $z' \circ [\gamma_i] = 0$ for any $i = 1, \ldots, m$ and therefore for any $i = 1, \ldots, M(\omega)$. By Theorem 5, $z' \in i_*H_1(\Delta)$. $\square$

4 Criterion and a necessary condition

Consider the map $[\omega] : H_1(M) \to \mathbb{R}$, $[\omega](z) = \int_z \omega$. Define $\operatorname{rk}\omega \stackrel{\text{def}}{=} \operatorname{rk}\operatorname{im}[\omega]$; obviously, $\operatorname{rk}\ker[\omega] + \operatorname{rk}\omega = \beta_1(M)$, the Betti number.

For a subgroup $H \subseteq H_{n-1}(M)$, denote $H^\ddagger \subseteq H_1(M)$ the subgroup $H^\ddagger = \{z \in H_1(M) \,|\, z \circ H = 0\}$. Note that $H_1 \subseteq H_2$ implies $H_2^\ddagger \subseteq H_1^\ddagger$.

Theorem 7 *$\mathcal{F}_\omega$ has a minimal component iff $H_\omega^\ddagger \not\subseteq \ker[\omega]$.*

PROOF. Suppose $\mathcal{F}_\omega$ has no minimal components, so that (2) is reduced to

$$\Delta = \Big(\bigcup_{i=1}^{K(\omega)} \gamma_i^0\Big) \cup \operatorname{Sing}\omega.$$

By Theorem 5, $H_\omega^\ddagger = i_* H_1(\Delta)$. Since $\int_z \omega = 0$ for any $z \in i_* H_1(\Delta)$, we have $H_\omega^\ddagger \subseteq \ker[\omega]$.

Suppose now $\mathcal{F}_\omega$ has a minimal component A. Consider $p \in A$ and the leaf $\gamma_p \ni p$. Through this point, in some its neighborhood $V_p \subseteq A$ a (local) integral curve $\varphi \subset A$ of the vector field ξ, $\omega(\xi) = 1$, can be drawn. Since φ is transverse to the leaves and the leaf γ_p is dense in A, there exists a point $q \in \gamma_p \cap \varphi$, $q \neq p$. Let $I \subset V_p \subseteq A$ be the segment of the integral curve between the points p and q. The leaf γ_p is connected, therefore there exists a curve $J \subset \gamma_p$ joining the points p and q. Then $c = I \cup J \subset A$ is a closed curve and $\int_c \omega = \int_I \omega \neq 0$. Since $[c] \circ H_\omega = 0$, we have $H_\omega^\ddagger \not\subseteq \ker[\omega]$. $\square$

This implies a necessary condition for $\mathcal{F}_\omega$ to have a minimal component:

Theorem 8 *If $\mathcal{F}_\omega$ has a minimal component then for any set of compact leaves $\gamma_1, \dots, \gamma_s \in \mathcal{F}_\omega$ it holds*

$$\langle [\gamma_1], \dots, [\gamma_s] \rangle^\ddagger \not\subseteq \ker[\omega].$$

Example 9 ([6]) *If a Morse form foliation on M_g^2 has g homologically independent compact leaves then it has no minimal components. Indeed, choose $[\gamma_1], \dots, [\gamma_g], D[\gamma_1], \dots, D[\gamma_g]$ (dual 1-cycles) as a basis of $H_1(M_g^2)$. Let $H = \langle [\gamma_1], \dots, [\gamma_g] \rangle$. Since $[\gamma_i] \circ D[\gamma_j] = \delta_{ij}$, $H^\ddagger = H$. Obviously, $H \subseteq \ker[\omega]$. By Theorem 8 the foliation has no minimal components.*

5 Sufficient conditions

We call a subgroup $H \subseteq H^1(M, \mathbb{Z})$ isotropic if $u \smile u' = 0$ (cup-product) for any $u, u' \in H$.

Definition 10 *A subgroup $G \subseteq H_1(M)$ is called parallel if there exists an isotropic subgroup $H \subseteq H^1(M, \mathbb{Z})$ such that any homomorphism $\varphi : G \to \mathbb{Z}$ is realized by an element of H, i.e. there exists $u \in H$ such that $u|_G = \varphi$.*

Theorem 11 *If any of the following equivalent conditions holds then $\mathcal{F}_\omega$ has a minimal component:*

(i) *For any parallel subgroup G it holds*

$$\operatorname{rk} G - \operatorname{rk}(G \cap \ker[\omega]) < \operatorname{rk} \omega; \tag{4}$$

(ii) *Inequality (4) holds for any parallel subgroup G such that $G \cap \ker[\omega] = 0$;*

(iii) *Inequality (4) holds for any maximal parallel subgroup G.*

Note that non-strict inequality in (4) holds for any subgroup G and any map $[\omega]$ out of general group-theoretic considerations.

PROOF. Condition (i) implies existence of a minimal component. Indeed, suppose $\mathcal{F}_\omega$ has no minimal components. Consider a group $G = DH_\omega = \langle D[\gamma_1], \dots, D[\gamma_m]\rangle$, where $[\gamma_1], \dots, [\gamma_m]$ is a basis in H_ω. By Theorem 6, $\operatorname{rk}\omega = \operatorname{rk} G - \operatorname{rk}(G \cap \ker[\omega])$. However, $G = DH_\omega$ is parallel. Indeed, associate with $Hom(DH_\omega, \mathbb{Z})$ the subgroup $H \subseteq H^1(M, \mathbb{Z})$, $H = \langle u_1, \dots, u_m\rangle$, where $u_i(z) = [\gamma_i] \circ z$. Let $\mathcal{D} : H^1(M, \mathbb{Z}) \to H_{n-1}(M)$ be Poincaré duality map. Then $\mathcal{D}(u_i \smile u_j) = \mathcal{D}u_i \circ \mathcal{D}u_j = [\gamma_i] \circ [\gamma_j] = [\gamma_i \bigcap \gamma_j] = 0$ since $\gamma_i \bigcap \gamma_j = \emptyset$ for $i \neq j$; thus H is isotropic.

(ii) $\Rightarrow$ (i). Let G be a parallel subgroup; $G = G' \oplus (G \cap \ker[\omega])$ for some (parallel) G'; then $\operatorname{rk} G - \operatorname{rk}(G \cap \ker[\omega]) = \operatorname{rk} G' < \operatorname{rk}\omega$ by (ii).

(iii) $\Rightarrow$ (ii). Let G be a parallel subgroup, $G \cap \ker[\omega] = 0$. For a maximal parallel subgroup $H \supseteq G$, choose $H' \supseteq G$ such that $H = H' \oplus (H \cap \ker[\omega])$. Then $\operatorname{rk} G \le \operatorname{rk} H' = \operatorname{rk} H - \operatorname{rk}(H \cap \ker[\omega]) < \operatorname{rk}\omega$ by (iii). □

Example 12 *Let $M = T_1^3 \# T_2^3$ (3-dimensional tori), $\operatorname{rk}\omega = 2$, and $\ker[\omega] \supseteq H_1(T_2^3)$. For any parallel subgroup G such that $G \cap \ker[\omega] = 0$ it holds $\operatorname{rk} G = 1$. By Theorem 11 (ii), $\mathcal{F}_\omega$ has a minimal component.*

The following Theorem 13 gives a sufficient condition simpler and more practical, though rougher, than Theorem 11.

Theorem 13 *Let $h(M)$ be the maximum rank of an isotropic subgroup in $H^1(M, \mathbb{Z})$. If $\operatorname{rk}\omega > h(M)$ then $\mathcal{F}_\omega$ has a minimal component.*

PROOF. Since for any parallel subgroup H it holds $\operatorname{rk} H \le h(M)$, the theorem follows from Theorem 11 (i). □

Remark 14 *Some methods of calculating $h(M)$ in terms of Betti numbers β_1 and β_2 can be found in [7], for instance:*

(i) *For $r = \operatorname{rk}\ker \smile$ (cup-product $H^1(M, \mathbb{Z}) \times H^1(M, \mathbb{Z}) \to H^2(M, \mathbb{Z})$),*

$$\frac{\beta_1 + \beta_2 r}{\beta_2 + 1} \le h(M) \le \frac{\beta_1\beta_2 + r}{\beta_2 + 1}.$$

In particular, if $\beta_2 = 1$ then $h(M) = \frac{1}{2}(\beta_1 + r)$; if $r = \beta_1$ then $h(M) = \beta_1$;

(ii) *If $\smile$ is surjective, then*

$$h(M) \le r + \frac{1}{2} + \sqrt{\left(\beta_1 - r - \frac{1}{2}\right)^2 - 2\beta_2};$$

(iii) *For the product,*

$$h(M_1 \times M_2) = \max\{h(M_1), h(M_2)\};$$

(iv) *For the connected sum with* $\dim M_i \geq 2$*,*

$$h(M_1 \# M_2) = h(M_1) + h(M_2).$$

Example 15 *For a torus* T^n *it holds* $h(T^n) = 1$ *and* $\operatorname{rk}\omega \leq n$*. The foliation has a minimal component if (Theorem 13) and only if [9]* $\operatorname{rk}\omega > 1$*.*

On a torus, $\operatorname{rk}\omega$ characterizes the topology of the foliation. This is, though, not always the case:

Example 16 *For* M_g^2 *it holds* $h(M_g^2) = g$ *and* $\operatorname{rk}\omega \leq 2g$*. The foliation has no minimal components if* $\operatorname{rk}\omega \leq 1$ *[9] and has a minimal component if* $g < \operatorname{rk}\omega \leq 2g$ *(Theorem 13). However, if* $2 \leq \operatorname{rk}\omega \leq g$*, the topology of the foliation may be quite different even in the same cohomology class. For instance, while in any cohomology class with* $\operatorname{rk}\omega \geq 2$ *there exists a form with minimal foliation [1], for any* $1 \leq \operatorname{rk}\omega \leq g$ *there exists* $\mathcal{F}_\omega$ *without minimal components.*

Indeed, consider g *tori* $T_i = M_i' \times S^1$*,* $M_i' = S^1$*, with a form* $\omega_i = \lambda_i\, dt$ *on* T_i*, where* t *is the coordinate along the* S^1*;* $\mathcal{F}_{\omega_i}$ *is compact. This form can be locally transformed into a form* ω_i' *with some spherical singularities. Using small spheres around these singularities, a connected sum* $M_g^2 = \#_{i=1}^g T_i$ *can be constructed with* ω_i *smoothly pasted together into a form* ω *on* M_g^2*;* $1 \leq \operatorname{rk}\omega = \operatorname{rk}\{\lambda_i\} \leq g$ *and* $\mathcal{F}_\omega$ *has no minimal components.*

Consider a Morse form in general position, i.e., with all periods being incommensurable; $\operatorname{rk}\omega = \beta_1(M)$. The foliation of such a form can have no minimal components: for example, if $\beta_1(M) = 0$ then all closed forms on M are exact. What is more, for any given $n \geq 3$ and $k \geq 0$ there exists a manifold M, $\dim M = n$ and $\beta_1(M) = k$, with a form ω in general position such that $\mathcal{F}_\omega$ has no minimal components:

Example 17 *The manifold* $M = \#_{i=1}^k M_i$ *and* ω *constructed as in Example 16 (*M_i *standing for* T_i *and* M *for* M_g^2*) with* $M_i' = S^{n-1}$ *and* $\operatorname{rk}\{\lambda_i\} = k$ *have the desired properties. Note that here* $\beta_2(M) = 0$*; however, by appropriate choice of* M_1'*,* $\beta_1(M_1') = 0$*, a similar example can be constructed for any given set of Betti numbers.*

Theorem 18 *Let* ω *be a Morse form in general position. If* $\smile\colon H^1(M,\mathbb{Z}) \times H^1(M,\mathbb{Z}) \to H^2(M,\mathbb{Z})$ *is non-trivial then* $\mathcal{F}_\omega$ *has a minimal component.*

PROOF. If $\smile$ is non-trivial then $h(M) < \beta_1(M) = \operatorname{rk}\omega$. By Theorem 13, $\mathcal{F}_\omega$ has a minimal component. $\square$

In addition, on M_g^2 all compact leaves of $\mathcal{F}_\omega$ with ω in general position are homologically trivial. Indeed, consider $[\gamma] = \sum n_i z_i$, where $\{z_i\}$ is the basis of cycles. Since $\int_\gamma \omega = \sum n_i \int_{z_i} \omega = 0$ and $\int_{z_i} \omega$ are incommensurable, all $n_i = 0$.

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